Faculty Working Papers

HARTLEY'S DEMAND-PRICE ANALYSIS

IN A CASE OF JOINT PRODUCTION

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In the October, 1971, issue of The Accounting Review, [1] Professor

Ronald V. Hartley developed several linear programming models of a joint cost

problem considered in managerial accounting textbooks. The problem is to assess

the desirability of processing joint products beyond their split-off point. An

important consideration in this problem is the possibility that the price charged

will influence the quantity of product taken by the market. Unfortunately, linear

programming models do not readily admit demand functions into their structure. The

result is that the optimal production schedule arising from such a model is con-

ditional on a particular set of prices and that the demand function must be

accommodated in a separate analysis which Hartley calls a "price-demand analysis." [2]

Unfortunately, the general form of the analysis is not completely specified; but,

more importantly, an alternative formulation of the problem would obviate the need

for such an analysis. This paper recommends reformulation of the joint cost problem

as a nonlinear programming problem [3] in which a demand function is given explicit

representation. The nonlinear model simultaneously determines the optimal price

and output policies and its application is less likely to lead to confusion and

error.

Hartley considers a case of joint production in which a single input, X,

leads to four outputs, A*, A', B*, and B'. Products A* and B* appear at the split-

off point after joint processing of the input X; products A' and B' result from

additional processing of A* and B* in separate facilities.

Hartley's LP models assume price is a given constant but treat the quantity demanded in two ways. In some models, the market will take any quantity supplied at the given price; in other models, the market will take any quantity supplied at the given price but only up to a fixed limit. In the first case there is no need to worry about inventories or about disposition of production in excess of sales. In the second case, however, the optimum production schedule may result in the production of one or more products in excess of the amount the market is willing to accept. For example, it may be profitable to produce and sell one product even though limits on demand preclude sale of a joint product that arises as a consequence of producing the first product.

The question is how the effect on profit of such overproduction can be represented in the decision model. Hartley's LP models assume that overproduction will be disposed of at no net cost. But he notes that other possibilities exist. The disposition might have a nonzero effect on profit which would require alteration of the objective function and probably the constraints as well. Changes of this type can probably be accomplished within a linear programming model.

Hartley notes another case that cannot be completely accommodated by a linear model. It is the case in which all or part of the excess production will be taken by the market if the price on all units of that product is lowered. He proposes a price-demand analysis on the linear model. This analysis leads to correct decisions if carefully applied and may be more readily accepted in practice by those more familiar with linear than nonlinear programming models, but its application is awkward and incompletely specified by Hartley's paper.

Hartley considers an example $\underline{4/}$ in which the optimal production schedule results in an output of 115,000 units of intermediate joint product, A*, at the $8.00 price included in the LP model. Hartley recommends that the firm consider lowering the price of A* in order to increase the quantity of A demanded. He notes that sensitivity analysis indicates that the price of A* "could be reduced to zero without affecting the optimal solution to the formulated problem." $\underline{5/}$ In other words, profit could be reduced by selling the 40,000 units of A* at any nonnegative price below $8.00 without changing the optimal production schedule. But if price is reduced, demand will rise. Depending on the elasticity of demand, the revenue from all units sold at the lower price may be larger, smaller, or the same as revenue from the 40,000 units sold at the $8.00 price. The task is to determine the price between zero and $8.00 that maximizes revenue from Product A*. If that price leads to sales of 115,000 units or less, then the optimal price-output policy is to sell A* at the revenue-maximizing price and to following the production schedule initially determined. If the revenue-maximizing price leads to sales of more than 115,000 units, then "the model will have to be revised and rerun" provided the revenue-maximizing price of A* "exceeds the contribution to be gained from further pricessing." $\underline{7/}$ In other words, even if the revenue-maximizing price leads to sales of A* in excess of 115,000 units, the intially determined production schedule remains optimal unless profit is increased more by selling a unit of A* than by turning it into Product A'. This assumes that the optimal production schedule fully utilizes capacity to produce A*. The assumption is satisfied in the numerical example Hartley considers, but it is

not necessarily satisfied in other cases. If A* can be produced without reducing production of A', then such excess capacity should be fully utilized and A* sold at whatever positive price. This action, of course, requires a departure from the production schedule originally determined.

The point is that it is awkward to take demand functions into an analysis in which they are present from the first. The analysis is clearer if the relevant demand functions are included in the programming model. This proposition can be demonstrated on Hartley's Case 3. Assume that the demand function for Product A is linear and of the form $a + bx_1$, where $a > 0$ and $b < 0$. Hartley's formulation of Case 3 requires only two modifications. First, the term of the objective function giving the contribution margin of sales of Product A* would be changed from $8x_1$, where 8 is the constant market price initially assumed by the LP model and x_1 is the quantity sold, to $(a + bx_1)x_1$ which introduces a quadratic term in the objective function. Second, the market constraint of 40,000 units on Product A* would be deleted from the program. The resulting quadratic programming model is given in Table 1. Since the quadratic portion of the objective function is negative definite, the program has a unique, global solution[8] which will specify the optimal price-output policy without requiring further analysis.

TABLE 1. —Quadratic Programming Problem for Hartley's Case 3

MAXIMIZE:

$$(a + bx_1)x_1 + 9x_2 + 7x_3 + 6x_4 - 2x_5 + 0x_6 + 0x_7 \qquad (8')$$

SUBJECT TO:

$$x_6 + x_7 + x_8 = 5{,}000 \qquad (9)$$

$$x_1 + x_2 - x_6 + x_{10} = 0 \qquad (11)$$

$$x_3 + x_4 - x_7 + x_{11} = 0 \qquad (12)$$

$$-3x_5 + x_6 + x_{12} = 0 \qquad (13)$$

$$3x_5 - x_6 + x_{13} = 0 \qquad (13)$$

$$-2x_5 + x_7 + x_{14} = 0 \qquad (14)$$

$$2x_5 - x_7 + x_{15} = 0 \qquad (14)$$

$$3x_2 + x_4 + 1.5x_5 + x_{16} = 80{,}000 \qquad (15)$$

$$3x_2 + x_5 + x_{17} = 40{,}000 \qquad (16)$$

and $x_i \geqslant 0,\; i = 1, 2, \ldots, 8, 10, 11, \ldots, 17.$

F O O T N O T E S

1/ Ronald V. Hartley, "Decision Making When Joint Products are Involved," The Accounting Review, XLVI (October, 1971), 746-755.

2/ Ibid., p. 751.

3/ The formulation recommended here is a special case of a nonlinear programming problem considered in another context by Roman L. Weil, Jr. Weil's formulates th problem along lines suggested by Professor Bierman.
See Roman L. Weil, "Allocating Joint Costs," The American Economic Review, LVIII (December, 1968), 1342-1345, and Harold Bierman, Jr., "Inventory Valuation: The Use of Market Prices," The Accounting Review, XLII (October, 1967), 731-737.

4/ This example is Hartley's Case 3 which encompasses the fundamental elements of the other cases considered by his paper.

5/ Hartley (1971), p. 751.

6/ This amounts to accepting the truncation of the demand function at the $8.00 price. In some cases, the firm's situation might be improved by considering prices over $8.00. For example, it is possible that demand for A* may be so inelastic that profits would actually be improved by calculating the LP model for a higher price of A* and a lower market limit. The quadratic programming formulation suggested below does not require that demand be truncated and thereby precludes the error of ignorning this possibility.

7/ Hartley (1971), p. 751.

8/ See Hadley's Chapter 7 for demonstration of the solution's existence for a description of algorithms by which it can be found. G. Hadley, Nonlinear and Dynamic Programming, Reading, Massachusetts: Addison-Wesley Publishing Comapny, Inc., 1964, pp. 212-240.